AXIAL FLOW OF A NONLINEAR VISCOPLASTIC FLUID THROUGH CYLINDRICAL PIPES

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The differential equation of motion is derived for fluids obeying the Bulkley–Herschel law and flowing through pipes of arbitrary cross section.

It has been pointed out in several studies [1, 2] that the Shvedov–Bingham model is inadequate for describing many materials with both viscous and plastic characteristics. The authors here have, with the aid of a rotary viscometer, obtained nonlinear relations for the viscoplastic flow of polymer-cement composites which quite well agree with the triparametric Bulkley–Herschel equation [3]. In tensor form, this equation is

$$\Pi_0 = 2\left(\frac{\tau_0}{h} + kh^{n-1}\right)\dot{\Phi}_0. \tag{1}$$

A simultaneous solution of this and the Cauchy equation leads to a tensorial equation of motion [2]:

$$2\left(\frac{\tau_0}{h} + kh^{n-1}\right)\operatorname{div}\dot{\Phi}_0 + 2\left[-\frac{\tau_0}{h^2} + k(n-1)h^{n-2}\right]\operatorname{grad}h - \operatorname{grad}p = \rho_0\vec{a}. \tag{2}$$

We will consider the laminar steady flow of a fluid according to Eq. (1) through a pipe of arbitrary cross section and the axis in line with the z-axis of a Cartesian system of coordinates. We introduce the notation $x = x_1$, $y = x_2$, $z = x_3$. The stress tensor components along the coordinate axes are then

$$\tau_{ij} = -\delta_{ij}p + \left(\frac{\tau_0}{h} + kh^{n-1}\right)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{3}$$

with δ_{ij} denoting the Kronecker delta, $u_1 = u_x$, $u_2 = u_y$, $u_3 = u_z$.

Projecting Eq. (2), with (3) taken into consideration, on the coordinates and disregarding the inertia forces, we obtain

$$\sum_{j=1}^{3}\left[-\delta_{ij}\frac{\partial p}{\partial x_j} + \left(\frac{\tau_0}{h} + kh^{n-1}\right)\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right.$$
$$\left. + \left(-\frac{\tau_0}{h^2} + k(n-1)h^{n-2}\right)\frac{\partial h}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] = 0 \tag{4}$$
$$(i = 1,\ 2,\ 3).$$

Since the stream lines are parallel to the cylinder axis, hence the velocity components u_x and u_y are equal to zero, while $u_z = \varphi(x, y)$. Therefore, Eqs. (4) become

$$-\frac{\partial p}{\partial x} = 0; \quad -\frac{\partial p}{\partial y} = 0; \tag{5}$$

$$-\frac{\partial p}{\partial z} + \left(\frac{\tau_0}{h} + kh^{n-1}\right)\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) + \left[-\frac{\tau_0}{h^2} + k(n-1)h^{n-2}\right]\left[\frac{\partial h}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right) + \frac{\partial h}{\partial y}\left(\frac{\partial\varphi}{\partial y}\right)\right] = 0, \tag{6}$$

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where

$$h=\sqrt{\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2}.$$

The continuity equation is satisfied identically. It follows from Eqs. (5) and (6) that the pressure is a linear function of the z-coordinate and, therefore,

$$-\frac{\partial p}{\partial z}=\text{const}=\alpha>0.$$

Inserting the values of derivatives $\partial h/\partial x$, $\partial h/\partial y$ into Eq. (6) and performing a few elementary operations, we arrive at the Bulkley–Herschel differential equation of axial flow through cylindrical pipes of arbitrary cross section:

$$h^{-3}(\tau_0+kh^n)\left[\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial y^2}-2\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}\cdot\frac{\partial^2 u}{\partial x\partial y}+\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial x^2}\right]+knh^{n-3}\left[\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial x^2}+2\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}\cdot\frac{\partial^2 u}{\partial x\partial y}+\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}\right]+\alpha=0. \quad (7)$$

This equation cannot be solved for the general case. Let us change it to a form convenient for solving various specific problems related to the flow of such fluids through pipes and annular channels with cross sections having a constant radius of curvature.

Inside the cross section we define some region bounded by a closed curve $\varphi(x, y) = \text{const}$. These curves (family of velocity isolines) lie within the plastic-deformation zone and do not intersect. The quasi-solid core of the stream is bounded by the maximum-velocity isoline.

Let ν be the outer normal to a velocity isoline. Obviously, the intensity of shear rates with respect to the modulus is equal to the normal derivative of the flow velocity:

$$\frac{\partial\varphi}{\partial\nu}=\pm h. \quad (8)$$

The – sign in Eq. (8) applies to a solid cylinder, because function $\varphi(x, y)$ decreases along the normal.

According to [4], the curvature of a velocity isoline can be expressed as

$$\frac{1}{\rho}=-h^{-3}\left[\frac{\partial^2\varphi}{\partial x^2}\left(\frac{\partial\varphi}{\partial y}\right)^2-2\frac{\partial\varphi}{\partial x}\cdot\frac{\partial\varphi}{\partial y}\cdot\frac{\partial^2\varphi}{\partial x\partial y}+\frac{\partial^2\varphi}{\partial y^2}\left(\frac{\partial\varphi}{\partial x}\right)^2\right]. \quad (9)$$

Inserting (8), (9), and the second normal derivative of the velocity function $\partial^2\varphi/\partial\nu^2$ into Eq. (7) will reduce this equation to

$$knh^{n-1}\frac{\partial^2\varphi}{\partial\nu^2}-\frac{\left(\pm\tau_0+kh^{n-1}\frac{\partial\varphi}{\partial\nu}\right)}{\rho}+\alpha=0. \quad (10)$$

The sign before τ_0 in Eq. (10) is selected so as to agree with the sign of the normal derivative of velocity, because the shear stress must be greater than the yield point.

With $n = 1$, (10) becomes the same equation which has been derived in [4] for a Shvedov–Bingham fluid.

1. Special Cases. We consider the axial flow of a nonlinear viscoplastic fluid through a circular cylinder. In this case

$$u=\varphi(r);\quad \frac{\partial\varphi}{\partial\nu}=\frac{du}{dr}<0;\quad h=-\frac{du}{dr};\ \rho=-r,$$

and, therefore, Eq. (10) becomes

$$k\frac{d}{dr}\left|\frac{du}{dr}\right|^n+\frac{k}{r}\left|\frac{du}{dr}\right|^n=\alpha-\frac{\tau_0}{r}. \quad (11)$$

Integrating Eq. (11) and determining the constant of integration from the conditions

$$u(R) = 0 \text{ (condition of adhesion) and } du/dr = 0 \text{ at } r = r_0,$$

with the radius of the core $r_0 = 2\tau_0/\alpha$, we obtain the well-known equation [2] of the velocity profile:

$$u(r) = \frac{2kn}{\alpha(1+n)}\left[\left(\frac{\alpha R}{2k} - \frac{\tau_0}{k}\right)^{\frac{1+n}{n}} - \left(\frac{\alpha r}{2k} - \frac{\tau_0}{k}\right)^{\frac{1+n}{n}}\right].$$

2. Flow between Parallel Planes. The distance between parallel planes will be denoted by 2H. With the origin of coordinates located on the median plane and the z-axis running in the direction of flow, the x-axis will run along the outer normal to a velocity isoline and, therefore,

$$\frac{\partial\varphi}{\partial\nu} = \frac{du}{dx} < 0.$$

Since the velocity isolines are parallel to the boundary planes, hence $\rho = \infty$ and Eq. (10) becomes

$$\frac{d}{dx}\left(-\frac{du}{dx}\right)^n = \frac{\alpha}{k}. \tag{12}$$

Integrating Eq. (12) and considering that the system adheres to the boundary planes, with the thickness of the quasisolid core $2h_0 = 2\tau_0/\alpha$, we arrive at the following equation for the velocity profile:

$$u(x) = \frac{n}{(1+n)}\left(\frac{\alpha}{k}\right)^{1/n}\left[(H - \tau_0)^{\frac{1+n}{n}} - (x - \tau_0)^{\frac{1+n}{n}}\right].$$

The flow rate per unit channel width is

$$Q = 2h_0 u_{\max} + 2\int_{h_0}^{H} u(x)\,dx, \text{ where } u_{\max} = u(h_0).$$

NOTATION

h is the intensity of strain rates;
Π_0 is the deviator of the stress tensor;
Φ_0 is the deviator of the strain rate tensor;
τ_0 is the static yield point;
k is the consistency index;
n is the exponent of viscous anomaly;
u is the velocity;
α is the pressure drop;
ν is the outer normal to velocity isoline;
ρ is the radius of curvature;
r is the radial coordinate;
R is the pipe radius;
p is the pressure;
r_0 is the radius of quasisolid core.

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